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Convergence of Rational Interpolants with Preassigned Poles

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We study the following problem. Given a domain Ω containing infinity, is it possible to choose a sequence of polynomials Q_n, $n = 1, 2, ...$, where Q_n has degree n, so that the following condition holds: if a function f is analytic in Ω and P_n is the polynomial part of the Laurent expansion of $Q_n f$ at infinity, then P_n/Q_n converges to f, as n tends to infinity, uniformly on bounded closed subsets of Ω? We get a complete solution of this problem when Ω is regular for Dirichlet's problem. For irregular domains we obtain some results having independent interest but a main problem remains open: is it possible to find such polynomials Q_n for some irregular domains Ω? © 1997 Academic Press

1. INTRODUCTION

Consider the problem of uniform approximation in a domain Ω of an analytic function f in Ω where the function is given by its Taylor series at some point $z_0 \in \Omega$. For the sake of convenience we suppose that $z_0 = \infty$. Then the corresponding Taylor series has the form

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

and this series converges inside the maximal disk, centered at infinity, where f is analytic, and diverges outside this disk.

This incomplete form of convergence, which is probably the main disadvantage of Taylor series comes from the fact that the partial sums of Taylor

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series are polynomials and they are, in particular, not able to approximate analytic functions around their poles. To overcome this problem Padé approximants (PAs) were introduced. These are rational functions matching $f(z)$ at infinity with maximal possible degree; for the precise definition see for instance [1]. PAs are very good for uniform approximation of some special classes of functions but in general we can not guarantee convergence anywhere except at the point $z=\infty$ itself (see [11]). The cause for such unpredictable behaviour of PAs is mainly a chaotic, spurious distribution of their poles for large classes of functions.

The difficulties with spurious poles led to the definition of a type of rational approximants with preassigned poles, known also as Padé type approximants (PTAs; see Definition 1). PTAs provide uniform convergence for analytic functions f in a general class of domains Ω but we need some special information on Ω. The poles of PTAs P_n/Q_n are preassigned in the sense that we may choose the denominator Q_n ourselves as any polynomial of degree n. This gives a great freedom in determining these approximants. The main problem investigated in this paper is how to choose, for a given domain Ω, polynomials $\{Q_n\}_{n=0}^{\infty}$ providing convergence for any function analytic in Ω, and if such a choice is possible at all or not. If such polynomials exist we say that they provide convergence in the PTA problem for the domain Ω.

This problem has different solutions depending on whether the domain Ω is regular (with respect to the Dirichlet problem) or not. For the regular case we give several conditions (see Conditions (1)–(5) in Theorem 1) on a sequence of polynomials $\{Q_n\}$ each equivalent to the fact that this sequence provides convergence in the PTA problem. Corollary 1 and 2 give a concrete way of constructing such polynomials. In the irregular case the problem is much more complicated and Conditions (1)–(5) are no longer equivalent. We have established a hierarchy of these conditions (see Theorem 2 and Proposition 1 and 2 which we consider as the main contributions in this paper) except for one implication which remains open. This implication must answer the following important question: if Ω is irregular does there exist a sequence of polynomials providing convergence in the PTA problem? We think that such a sequence does not exist for any irregular domain Ω (see Conjecture 2 below).

In conclusion we note that Conditions (2)–(5) in Theorem 1 and 2 below are main characteristics usually studied in different problems on polynomial and rational approximation, and the establishment of a hierarchy of these conditions in the general case is a problem of independent interest.

In this paper we work with PTAs defined by means of interpolation at $z_0=\infty$. Assume that the set of zeros of $\{Q_n\}_0^{\infty}$ has no limit point in Ω. It then follows from Theorem 1 that the polynomials $\{Q_n\}$ provide convergence in the PTA problem for a regular domain Ω containing infinity if the asymptotic zero distribution of $\{Q_n\}$ is given by the equilibrium measure of the boundary $\partial\Omega$ of Ω. If instead we use PTAs corresponding to interpolation at any chosen point z_0 in the complex plane the polynomials $\{Q_n\}$ provide convergence in the PTA problem for a regular domain Ω containing z_0 if the asymptotic zero distribution of $\{Q_n\}$ is given by the harmonic measure of $\partial\Omega$ evaluated at z_0 (see [4]).

This paper is the fourth in a series of papers (see [1]–[3]) on PTAs by the authors.

2. DEFINITIONS AND NOTATION

We use the following notation:

$\bar{\mathbb{C}}$: The extended complex plane, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Ω: A domain (open, connected set) in $\bar{\mathbb{C}}$ containing infinity, $\infty \in \Omega$.

$\partial\Omega$: The boundary of Ω.

K: The complement of Ω, $K = \bar{\mathbb{C}} \setminus \Omega$. We always assume that K has positive logarithmic capacity, $\operatorname{cap}(K) > 0$.

qu.e.: Quasi everywhere, i.e., everywhere except on a set of logarithmic capacity zero.

f: A function analytic in Ω.

$g_\Omega(z)$: The Green function of Ω with pole at infinity.

$Q_n(z)$: A monic polynomial of degree n.

μ_n: The zero counting measure of Q_n; μ_n puts mass $1/n$ at each zero of Q_n counting multiplicity.

μ: A positive finite measure with compact support $S(\mu) \in K$.

$U(\mu, z)$: The logarithmic potential of μ, $U(\mu, z) = \int \log(1/|z-t|)\, d\mu(t)$.

$\mu_n \to \mu$: Weak star convergence of measures: $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$ for arbitrary continuous functions φ in $\bar{\mathbb{C}}$.

μ_K: The equilibrium measure of K.

Definition 1. Let $Q(z)$ be any polynomial, not identically zero, and denote by $P(z)$ the polynomial part of the Laurent expansion of $Q(z)\, f(z)$ at infinity. Then the rational function $\pi(z) = P(z)/Q(z)$ is called the *Padé type approximant* (PTA) of f (at infinity) with preassigned denominator $Q(z)$ (or PTA with preassigned poles at the zeros of $Q(z)$).

The definition means that

$$Q(z)\, f(z) - P(z) = \mathcal{O}(z^{-1}), \qquad \text{as} \quad z \to \infty.$$

Evidently, $\pi(z)$ does not depend on the normalization of $Q(z)$. We always assume that $Q(z)$ is a monic polynomial, i.e. the leading coefficient of Q is 1.

Let P_n/Q_n be the PTA of f with preassigned denominator Q_n. For the error of approximation Cauchy's integral theorem and integral formula give (see for instance [6]):

$$R_n(z) := f(z) - \frac{P_n(z)}{Q_n(z)} = \frac{-1}{2\pi i} \int_\gamma \frac{Q_n(t)}{Q_n(z)} \frac{f(t)}{t-z}\, dt, \tag{1}$$

for all points z outside γ, where $\gamma \subset \Omega$ is a simple closed positively oriented curve, or a finite union of such curves, winding once around each point of K.

Definition 2. We say that the sequence of polynomials $\{Q_n\}_0^\infty$ *provides convergence* in the PTA problem for a domain Ω if for an arbitrary function f analytic in Ω the corresponding PTAs converge to f uniformly on bounded closed subsets of Ω.

Since the polynomials $\{Q_n\}$ are going to serve as denominators of rational approximants to analytic functions in Ω, the following condition of nonvanishing of Q_n in Ω is natural.

Definition 3. We say that the sequence $\{Q_n\}_0^\infty$ has *asymptotically no zeros in* Ω if the set of zeros of the polynomials in the sequence has no limit point in Ω.

Definition 4. Let Ω and K be defined as above and let μ be a measure with support on K. The measure μ' is the *sweeping out* (balayage) of μ onto $\partial\Omega$ if $S(\mu') \subset \partial\Omega$, $\mu'(K) = \mu(K)$, and

$$U(\mu, z) = U(\mu', z) \qquad \text{for all} \quad z \in \Omega. \tag{2}$$

We note that according to the classical definition of sweeping out we should require in (2) equality quasi everywhere on $\bar{\Omega} = \Omega \cup \partial\Omega$ which, due to continuity of potentials, implies that (2) holds everywhere in Ω. However, the assumption that (2) holds everywhere in Ω implies that (2) holds also everywhere on $\partial\Omega$ and that the sweeping out of μ onto $\partial\Omega$ is unique (see [9], Chap. II, Theorem 4.7).

Definition 5. We say that the zeros of $\{Q_n\}_0^\infty$ have *regular asymptotic distribution* on K if every weak limit of the set of unit measures $\{\mu_n\}_1^\infty$ is supported by K and its sweeping out onto $\partial\Omega = \partial K$ coincides with the equilibrium measure μ_K. The stronger property, when $\mu_n \to \mu_K$, we refer to as *equilibrium asymptotic distribution.*

DEFINITION 6. Let Q_n be the nth monic orthogonal polynomial with respect to μ. The measure μ is *regular* if for the polynomials $\{Q_n\}$ the limit relation (3) in Theorem 1 below holds locally uniformly outside the convex hull of the support $S(\mu)$, where Ω is the unbounded component of the complement of $S(\mu)$ and $K = \bar{\mathbb{C}} \setminus \Omega$.

3. RESULTS

We start with the case of regular domains.

THEOREM 1. *If Ω is a regular domain in $\bar{\mathbb{C}}$ containing infinity, and $\{Q_n(z) = z^n + \cdots\}_{n=0}^{\infty}$ is a sequence of polynomials having asymptotically no zeros in Ω, the following conditions are pairwise equivalent:*

(1) *$\{Q_n\}_0^{\infty}$ provides convergence in the PTA problem for Ω.*

(2) $\lim_{n\to\infty} \max_{z\in K} |Q_n(z)|^{1/n} = \mathrm{cap}(K)$, *where* $K = \bar{\mathbb{C}} \setminus \Omega$.

(3) $\lim_{n\to\infty} |Q_n(z)|^{1/n} = \mathrm{cap}(K)\, e^{g_\Omega(z)}$, *uniformly on bounded closed subsets of Ω.*

(4) *The zeros of $\{Q_n\}$ have regular asymptotic distribution.*

(5) *For an arbitrary bounded closed set $F \subset \Omega$ there exists an open neighbourhood O of K such that*

$$\lim_{n\to\infty} \left(\sup \left\{ \left| \frac{Q_n(t)}{Q_n(z)} \right|, t \in O, z \in F \right\} \right) = 0.$$

Many of these implications are more or less well-known. Condition (1) follows from Condition (5) and the error formula (1). For the equivalence (2) $\Leftrightarrow$ (3), see [12], Section 7.4 or [8], Theorem 2.5, and for the implication (3) $\Rightarrow$ (1), see [12], Section 8.4, IIIb.

In the general case, when Ω is not necessarily regular, the equivalence problem of the Conditions (1)–(5) is much more complicated and not completely solved. The conditions are no longer equivalent but we have the following result.

THEOREM 2. *Let Ω be any domain in $\bar{\mathbb{C}}$ containing infinity, with $K = \bar{\mathbb{C}} \setminus \Omega$ and* $\mathrm{cap}(K) > 0$, *and let $\{Q_n(z) = z^n + \cdots\}_0^{\infty}$ be a sequence of polynomials having asymptotically no zeros in Ω. Then the following implications hold between Conditions* (1)–(5) *of Theorem* 1:

$$(5) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4).$$

Concerning Conditions (5) and (1) we make the following conjecture.

CONJECTURE 1. *Under the hypothesis of Theorem* 2 *we have* (1) $\Rightarrow$ (5).

Below we prove (Section 5.4) that this conjecture is equivalent to the following more important problem.

CONJECTURE 2. *For the existence of polynomials* $\{Q_n\}_0^\infty$ *providing convergence in the PTA problem for a domain* Ω *defined in Theorem* 2, *it is necessary and sufficient that* Ω *is regular.*

The existence of polynomials providing convergence in the case of regular domains Ω follows from Theorem 1 since the Fekete polynomials for K satisfy Condition (2). The substantial part of Conjecture 2 is that there does not exist polynomials providing convergence for irregular domains. Below we prove (see Section 5.5) the validity of this assertion for the case when $K=\overline{\mathbb{C}}\backslash\Omega$ has an isolated point and the polynomials Q_n have no zeros in Ω. We note that even in this case the proof we have is quite complicated. To have a convenient reference we formulate the result in the following proposition.

PROPOSITION 1. *If* Ω *is a domain containing infinity and such that* $K=\overline{\mathbb{C}}\backslash\Omega$, $\operatorname{cap}(K)>0$, *has an isolated point, then there is no sequence of polynomials* $\{Q_n\}_0^\infty$ *non-vanishing in* Ω *and providing convergence in the PTA problem for* Ω.

In contrast to Theorem 1, in Theorem 2 we do not claim that we have the following three implications: (5) $\Leftarrow$ (1), (1) $\Leftarrow$ (2), and (2) $\Leftarrow$ (3). We have discussed the first of these implications in Conjecture 1. We discuss the other two in our next proposition which we prove in Section 5.3.

PROPOSITION 2. *The implication* (2) $\Leftarrow$ (3) *is true if and only if* Ω *is regular. The implication* (1) $\Leftarrow$ (2) *is not true in general for irregular domains* Ω.

We do not know if the implication (1) $\Leftarrow$ (2) is false for all irregular domains.

To summarize the above discussion we see that the property of providing convergence in the PTA problem for a domain Ω, puts quite strong restrictions on a sequence $\{Q_n\}_0^\infty$ and on Ω as well. In particular, this condition is stronger than Conditions (2), (3) and (4) of Theorem 1, which are the most common criteria for regular behaviour of polynomials, and it may be equivalent (Conjecture 1) to Condition (5).

As for the practical choice of polynomials providing convergence in the case of regular Ω we may use different types of extremal polynomials. We state below (Corollary 1 and 2) that orthogonal polynomials generated by some regular measure provide convergence. We prove this in Section 5.6.

Let μ be a regular measure with compact support $S(\mu)$ such that the unbounded component of the complement of $S(\mu)$ coincides with Ω. This means that $\partial\Omega \subset S(\mu) \subset K$. First we formulate the result when K is a convex set.

COROLLARY 1. *Let Ω be a domain containing infinity and $K = \overline{\mathbb{C}} \backslash \Omega$ a convex set. Then the monic orthogonal polynomials generated by a regular measure μ satisfying $\partial\Omega \subset S(\mu) \subset K$, provide convergence in the PTA problem for the domain Ω.*

As concerns general regular domains Ω, they may contain zeros of the orthogonal polynomials. However, it is always possible to omit some of the zeros of the orthogonal polynomials to get new polynomials with asymptotically no zeros in Ω. In particular, we have the following more general version of Corollary 1.

COROLLARY 2. *Let Ω be a regular domain and μ a regular measure satisfying $\partial\Omega \subset S(\mu) \subset K$. Let V denote an open bounded set, $\bar{V} \subset \Omega$, where $\bar{V}$ is the closure of V. Denote by $z_{n1}, z_{n2}, ..., z_{nm}$, $m = m(n)$, the zeros belonging to $\bar{V}$ of the corresponding monic orthogonal polynomials Q_n generated by μ. Then, for an arbitrary analytic function f in Ω, the PTAs with preassigned denominators $\tilde{Q}_n(z) := Q_n(z)/(z - z_{n1}) \cdots (z - z_{nm})$ converge to f uniformly on compact subsets of V.*

Remark 1. It is a beautiful fact (see [13] or [10]) that although $m(n)$, the number of zeros of Q_n in $\bar{V}$, depends on n, it is bounded in n.

Finally, in Section 5.7 we prove the following proposition.

PROPOSITION 3. *In general, in Corollary 2 we cannot omit the assumptions that the set Ω and the measure μ are regular.*

4. PRELIMINARY FACTS

Lower Envelope Theorem (see [7], Theorem 3.8 or [9], Appendix). Let μ and $\{\mu_n\}_1^\infty$ be measures with support in a compact set K of $\mathbf{C}$. If $\mu_n \to \mu$ and $z_n \to z_0$ then

$$U(\mu, z_0) = \liminf_{n \to \infty} U(\mu_n, z_n)$$

for quasi every $z_0 \in \mathbf{C}$.

In addition, if an open set V has no intersection with the compact set K then $U(\mu_n, z) \to U(\mu, z)$ uniformly on compact subsets of V.

Principle of Domination ([7], Theorem 1.27 or [9], Appendix). Suppose that μ and ν are probability measures and ν is of finite logarithmic energy. If $U(\mu, z) \geqslant U(\nu, z)$ holds ν-almost everywhere, then it holds everywhere.

C-Absolute Continuity of Equilibrium Measure ([7], Section II.1). $\mu_K(E) = 0$ for any set E such that $\operatorname{cap}(E) = 0$.

The fine topology on $\mathbf{C}$ is the weakest topology on $\mathbf{C}$ for which all superharmonic functions (in particular potentials) are continuous. If O is a connected open set then the boundary of O in the fine and Euclidean topologies coincide (see for instance [10], Appendix).

5. PROOFS

Without loss of generality in all proofs below we assume that $\operatorname{cap}(K) = 1$.

5.1. *Proof of Theorem* 2. We start by proving the implication (1) $\Rightarrow$ (2) which is the central part of Theorem 2. Let the polynomials $\{Q_n(z) = z^n + \cdots\}_{n=0}^{\infty}$ provide convergence and introduce

$$M_n = \max_K |Q_n(z)|. \tag{3}$$

We have to prove that

$$\lim_{n \to \infty} M_n^{1/n} = \operatorname{cap}(K) = 1. \tag{4}$$

We know that $\liminf_{n \to \infty} M_n^{1/n} \geqslant 1$, since we have the limit 1 for the Chebyshev polynomials which minimize M_n. Suppose on the contrary that we do not have (4), and denote by z_n the point where $|Q_n(z)|$ attains its maximum on K. Then without loss of generality we may assume that

$$\lim_{n \to \infty} |Q_n(z_n)|^{1/n} > \alpha > 1 \tag{5}$$

(otherwise we may take a subsequence). Consider the following function

$$f(z) = \sum_{m=1}^{\infty} \frac{a_m}{z - z_m},$$

where a_m is $1/m^2$, $-1/m^2$ or 0 according to the specification below. Obviously, this sum converges uniformly on compact subsets of Ω and consequently defines an analytic function in Ω. By applying the error formula (1) to this function we obtain

$$R_n(z) = \frac{-1}{2\pi i} \int_\gamma \frac{Q_n(t)}{Q_n(z)} \cdot \sum_{m=1}^{\infty} \frac{a_m}{t - z_m} \frac{dt}{t - z},$$

for all z outside γ, where $\gamma \subset \Omega$ is a simple closed curve, or a finite union of such curves, winding once around K. By integrating termwise and applying Cauchy's integral formula we obtain, since $Q_n(t)/(t-z)$ is analytic in t inside γ,

$$R_n(z) = f(z) - \frac{P_n(z)}{Q_n(z)} = -\sum_{m=1}^{\infty} \frac{a_m \cdot Q_n(z_m)}{Q_n(z) \cdot (z_m - z)} \tag{6}$$

outside γ and, due to the arbitrariness in the choice of γ, everywhere in Ω.

Now, by using (6) and the fast growth of $Q_n(z_n)$ (see (5)) we shall try to get divergence of $R_n(z)$ at some point $z \in \Omega$ where $Q_n(z)$ does not grow so fast, as n tends to infinity.

First we prove the following lemma.

LEMMA 1. *For an arbitrary $\beta > 1$ there exists a point $z' \in \Omega$ such that*

$$\liminf_{n \to \infty} |Q_n(z')|^{1/n} < \beta. \tag{7}$$

Proof of Lemma 1. Suppose on the contrary that

$$\liminf_{n \to \infty} |Q_n(z)|^{1/n} \geqslant \beta \tag{8}$$

for all $z \in \Omega$. We may suppose also that the corresponding zero counting unit measures μ_n of Q_n have a weak star limit σ supported by K. Then, for the corresponding potentials we get, since Q_n have asymptotically no zeros in Ω,

$$U(\sigma, z) = \liminf_{n \to \infty} U(\mu_n, z) = -\liminf_{n \to \infty} \log |Q_n(z)|^{1/n} \leqslant -\log \beta < 0$$

for all $z \in \Omega$. Due to the properties of the fine topology described in Section 4, we get that

$$U(\sigma, z) \leqslant -\log \beta < 0$$

everywhere on $\partial\Omega$.

Denote by σ' the sweeping out measure of σ onto $\partial\Omega$. Then, due to the equality of the corresponding potentials on $\partial\Omega$ (see Section 2), we also have

$$U(\sigma', z) \leqslant -\log \beta < 0$$

everywhere on $\partial\Omega$, and, in addition, $S(\sigma') \subset \partial\Omega$. But this inequality contradicts the extremal property of the equilibrium measure μ_K of the compact set K, which says that

$$0 = -\log \operatorname{cap}(K) = \sup_{S(\mu_K)} U(\mu_K, t) = \min \sup_{S(\nu)} U(\nu, z),$$

where the minimum is taken over all probability measures ν supported by K (see [7], Theorem 2.3 (ii)). Lemma 1 is proved.

Now we fix a point $z' \in \Omega$ satisfying (7) with $\beta < \alpha$, and give the rule for the choice of the coefficients a_m, $m = 1, 2, 3, \ldots$. As we have said above, we always take a_m to be $\pm 1/m^2$ or 0. First we define an increasing subsequence of indices m_i, $i = 1, 2, \ldots$, for which $a_{m_i} \neq 0$. By using (5) and (7) we choose m_1 as the smallest integer for which

$$|Q_{m_1}(z_{m_1})|^{1/m_1} > \alpha \qquad \text{and} \qquad |Q_{m_1}(z')|^{1/m_1} < \beta.$$

After having defined m_i we choose m_{i+1} by the following conditions

$$|Q_{m_{i+1}}(z_{m_{i+1}})|^{1/m_{i+1}} > \alpha, \tag{9}$$

$$|Q_{m_{i+1}}(z')|^{1/m_{i+1}} < \beta, \tag{10}$$

$$\sum_{m = m_{i+1}}^{\infty} \frac{|Q_{m_i}(z_m)|}{m^2} < 1. \tag{11}$$

In this way we define an infinite sequence m_i, $i = 1, 2, \ldots$. For all indices m not belonging to this subsequence we put $a_m = 0$. Now we define the numbers a_{m_i}. We have already defined them up to the sign: $|a_{m_i}| = 1/m_i^2$. Having defined signs of $a_{m_1}, a_{m_2}, \ldots, a_{m_{i-1}}$ we define the sign of a_{m_i} from the following condition

$$\left| \frac{a_{m_i} \cdot Q_{m_i}(z_{m_i})}{z_{m_i} - z'} \right| \leqslant \left| \sum_{m=1}^{m_i} \frac{a_m Q_{m_i}(z_m)}{z_m - z'} \right|. \tag{12}$$

At least for one choice of the sign of a_{m_i} we have this inequality; here we use the fact that $\max\{|a+b|, |a-b|\} \geqslant |b|$.

Now we have defined all the coefficients a_m, $m=1, 2, ...,$ and we have to check the divergence of $R_n(z')$. We obtain

$$|R_{m_i}(z')| = \left|\frac{1}{Q_{m_i}(z')} \cdot \sum_{m=1}^{\infty} \frac{a_m Q_{m_i}(z_m)}{z_m - z'}\right|$$
$$\geqslant \frac{1}{|Q_{m_i}(z')|} \cdot \left(\left|\sum_{m=1}^{m_i} \frac{a_m Q_{mi}(z_m)}{z_m - z'}\right| - \left|\sum_{m>m_i} \frac{a_m Q_{m_i}(z_m)}{z_m - z'}\right|\right).$$

We get by (11), if $d = \operatorname{dist}(z', K)$,

$$\left|\sum_{m>m_i} \frac{a_m Q_{m_i}(z_m)}{z_m - z'}\right| \leqslant \frac{1}{d} \cdot \sum_{m=m_{i+1}}^{\infty} \left|\frac{Q_{m_i}(z_m)}{m^2}\right| \leqslant \frac{1}{d}. \tag{13}$$

From (12) and (9) we get, if $D = \max\{\operatorname{dist}(z', z), z \in K\}$,

$$\left|\sum_{m=1}^{m_i} \frac{a_m Q_{m_i}(z_m)}{z_m - z'}\right| \geqslant \left|\frac{a_{m_i} Q_{m_i}(z_{m_i})}{z_{m_i} - z'}\right| \geqslant \frac{\alpha^{m_i}}{m_i^2 \cdot D}. \tag{14}$$

Finally, (13), (10) and (14) give

$$|R_{m_i}(z')| \geqslant \frac{1}{\beta^{m_i}} \left(\frac{\alpha^{m_i}}{m_i^2 \cdot D} - \frac{1}{d}\right),$$

and, taking into account that $\alpha > \beta > 1$, we obtain

$$\lim_{i \to \infty} |R_{m_i}(z')| = \infty.$$

The implication $(1) \Rightarrow (2)$ is proved.

Proof of $(2) \Rightarrow (4)$ *in Theorem* 2. From Condition (2), with the assumption cap $K = 1$, follows that we may choose a sequence of positive numbers $\varepsilon_n \to 0$ such that

$$U(\mu_n, z) = \frac{-1}{n} \log |Q_n(z)| > -\varepsilon_n, \tag{15}$$

for all $z \in K$. Let us choose a weakly convergent subsequence of the zero counting measures $\{\mu_n\}, \mu_{n_i} \to \mu$. Then $S(\mu) \subset K$. By using the Lower envelope theorem (Section 4) and (15) we obtain that $U(\mu, z) \geqslant 0$ qu.e. on K. For the equilibrium measure μ_K we know that $U(\mu_K, z) = 0$ qu.e. on K. Hence, we get

$$U(\mu, z) \geqslant U(\mu_k, z) \tag{16}$$

qu.e. on K. If E is the exceptional set in (16), then $\mu_K(E)=0$ since cap $E=0$ (Section 4). This means that (16) holds μ_K-almost everywhere as well. Hence, the Principle of domination (Section 4) yields that

$$U(\mu, z) \geqslant U(\mu_K, z) \qquad \text{everywhere in } \mathbf{C}. \tag{17}$$

Suppose that for some point $z_0 \in \Omega$ we have strict inequality in (17),

$$U(\mu, z_0) > U(\mu_K, z_0). \tag{18}$$

Let $\gamma \subset \Omega$ be a simple closed curve passing through z_0 such that K is inside γ. Denote by μ_γ the equilibrium measure of γ. Then $U(\mu_\gamma, z) \equiv c_\gamma$ on and inside γ, where c_γ is some constant. By using Fubini's theorem we get

$$\int U(\mu, z)\, d\mu_\gamma = \int U(\mu_\gamma, z)\, d\mu = c_\gamma = \int U(\mu_\gamma, z)\, d\mu_K = \int U(\mu_K, z)\, d\mu_\gamma.$$

Consequently,

$$\int U(\mu, z)\, d\mu_\gamma = \int U(\mu_K, z)\, d\mu_\gamma. \tag{19}$$

However, by (17) and (18), and the continuity of the potentials $U(\mu, z)$ and $U(\mu_K, z)$ on γ, we should have strict inequality instead of equality in (19). This contradiction shows that $U(\mu, z) \equiv U(\mu_K, z)$ in Ω and, by definition, μ_K is the sweeping out of μ onto $\partial\Omega$.

Proof of (3) $\Rightarrow$ (4) *in Theorem* 2. Let μ be a weak star limit of a subsequence μ_{n_i} of the zero counting measures of Q_n, $\mu_{n_i} \to \mu$. Then, due to Condition 3 we have, since Q_{n_i} have asymptotically no zeros in Ω,

$$U(\mu, z) = \lim_{i \to \infty} U(\mu_{n_i}, z) = -\lim_{i \to \infty} \log |Q_{n_i}(z)|^{1/n_i} = -g_\Omega(z) = U(\mu_K, z),$$

for all $z \in \Omega$. By definition this means that the sweeping out of μ is μ_K.

Proof of (4) $\Rightarrow$ (3) *in Theorem* 2. Suppose that (4) holds but not (3). Then we easily get a contradiction by using the fact that from any subsequence of $\{\mu_n\}$ we may extract another subsequence which has a weak star limit.

Proof of (5) $\Rightarrow$ (1) *in Theorem* 2. This implication immediately follows from the error formula (1) if we choose γ in O. Theorem 2 is proved.

5.2. *Proof of Theorem* 1. After having proved Theorem 2 it remains to prove that (3) $\Rightarrow$ (5) for regular domains Ω. However, this implication follows if we use that for regular domains Ω the Green function is continuous on $\bar{\Omega}$, identically zero on $\partial\Omega$ and strictly positive in Ω. Theorem 1 is proved.

5.3. *Proof of Proposition* 2.

5.3.1. We start with the proof that if Ω is any irregular domain then Condition (3) does not imply Condition (2).

Condition (3) is equivalent to Condition (4). Consequently, what we have to do is to construct a sequence of monic polynomials $\{Q_n\}$ whose zeros have regular asymptotic distribution (in our case below we even have equilibrium asymptotic distribution), but

$$\limsup_{n\to\infty} (\max_{z\in K} |Q_n(z)|^{1/n}) > \operatorname{cap}(K) = 1.$$

Let z_0 be an irregular point of K. Then $g_\Omega(z_0) > \alpha > 0$. We claim that there exists a sequence of disks $\{O_m\}_1^\infty$ centered at z_0, so that their radii tend to zero and the boundaries ∂O_m do not contain points of K. This follows from Wiener's criterion for irregular points ([7], Chapter V) combined with an argument on a projection on a half-line from the irregular point and an application of the transfinite diameter. We omit the details.

Introduce $K_m = K\setminus O_m$ and let g_m be the Green function of $\Omega_m := \bar{\mathbb{C}}\setminus K_m$. Since $\Omega_m \supset \Omega$ we have $g_m(z) \geqslant g_\Omega(z)$ for all $z \in \Omega$. In particular, this inequality holds on ∂O_m for arbitrary m. Since g_m is harmonic and g_Ω subharmonic in O_m we get the same inequality at z_0 as well,

$$g_m(z_0) \geqslant g_\Omega(z_0) > \alpha > 0, \qquad m = 1, 2, \dots . \tag{20}$$

Fix m and denote by $\{P_n^{(m)}\}_{n=1}^\infty$ a sequence of monic polynomials with zeros on K_m having equilibrium asymptotic distribution. Then, by (3) $\Leftrightarrow$ (4) in Theorem 2, we obtain

$$\lim_{n\to\infty} |P_n^{(m)}(z_0)|^{1/n} = \operatorname{cap}(K_m) \cdot e^{g_m(z_0)}. \tag{21}$$

From (20) and the fact that $\operatorname{cap}(K_m) \to \operatorname{cap}(K) = 1$ we get, for sufficiently large m,

$$\operatorname{cap}(K_m)\, e^{g_m(z_0)} > q \cdot \operatorname{cap}(K) = q, \tag{22}$$

for some fixed $q > 1$. We may assume that this inequality holds for all m. Now, by (21) and (22), for any fixed m and sufficiently large n, $n > N(m)$, we obtain

$$|P_n^{(m)}(z_0)|^{1/n} > q \cdot \operatorname{cap}(K). \tag{23}$$

We extract a "diagonal" sequence $n(m)$ such that $n(m) > N(m)$ and $n(m+1) > n(m)$, and consider the sequence of polynomials $P^{(m)}_{n(m)}$, $m = 1, 2, \ldots$ with zero distributions $\mu_{n(m)}$. We may also guarantee that the zeros of these polynomials have equilibrium asymptotic distribution. In fact, first, fixing any closed disk $F \subset \Omega$, we can guarantee that $U(\mu_{n(m)}, z) \to U(\mu_K, z)$ uniformly on F. Then, for any weak star limit μ of the set of measures $\{\mu_{n(m)}\}$, we will have $U(\mu, z) = U(\mu_K, z)$ on F and, due to the uniqueness theorem for harmonic functions, everywhere in Ω as well. In addition, evidently $S(\mu) \in \partial\Omega$, and so $\mu = \mu_K$. We conclude that $\mu_{n(m)} \to \mu_K$. Consequently, for the polynomials $P^{(m)}_{n(m)}$ we have Condition (4) but, according to (23) we do not have Condition 2. Polynomials for missing indices can be chosen arbitrarily with zeros on $\partial\Omega$ having equilibrium asymptotic distribution.

5.3.2. Now we prove that for irregular domains Ω Condition (2) does not, in general, guarantee Condition (1).

Let $\{T_n(z) = z^n + \cdots\}_0^\infty$ be the Chebyshev polynomials for $\Delta = [-2, 2]$. It is well-known that

$$\max_{z \in \Delta} |T_n(z)|^{1/n} \to \operatorname{cap}(\Delta) = 1, \qquad \text{as} \quad n \to \infty.$$

Consider the compact set $K = \Delta \cup \{3\}$, and $\Omega = \overline{\mathbb{C}} \backslash K$. Put $Q_n(z) = T_{n-1}(z) \cdot (z-3)$, $n = 1, 2, \ldots$. Evidently

$$\lim_{n \to \infty} \max_{z \in K} |Q_n(z)|^{1/n} \to \operatorname{cap}(K) = 1,$$

i.e., the polynomials $\{Q_n\}$ satisfy Condition (2) for the compact set K. Let us check that Condition (1) is not satisfied. Consider $f(z) = 1/(z-3)^2$. If the corresponding PTAs $P_n(z)/Q_n(z)$ approximate $f(z)$ uniformly in Ω, as $n \to \infty$, the rational functions $P_n(z) \cdot (z-3)/Q_n(z)$, which are analytic in $\Omega \cup \{3\}$, approach the function $1/(z-3)$. But this is impossible which we see by expressing, for example, the residue of the function $1/(z-3)$ at the point $z = 3$ by the integral on the circle $|z-3| = 1/2$.

5.4. Now we prove that Conjecture 1 is equivalent to Conjecture 2.

If Conjecture 2 is true and we have Condition (1), this means that Ω is regular and by Theorem 1 we have Condition (5) as well.

The other direction is more complicated. Suppose that Conjecture 1 is true. The fact that for arbitrary regular Ω there exist polynomials satisfying Condition (1) is trivial; take for example Fekete polynomials, or see Corollary 2 above. What we need to prove is that if Ω is irregular then no sequence of polynomials satisfies Condition (1). Suppose on the contrary that Ω is irregular and that the polynomials $\{Q_n\}$ satisfy Condition (1).

Then, due to the assumption that Conjecture 1 is true, they also satisfy Condition (5), and, by Theorem 2, all the other conditions (2)–(4). Now we prove that for irregular domains Condition (5) never holds.

Let $z_0 \in \partial\Omega$ be an irregular point of K. Then, by the definition of irregular points, $e^{g_\Omega(z_0)} > q > 1$. Due to properties of the fine topology described in Section 4, there exists a point $z_1 \in \Omega$ arbitrarily close to z_0 satisfying the same inequality

$$e^{g_\Omega(z_1)} > q > 1. \tag{24}$$

Using again the same argument about the fine topology, but starting with some regular point $\omega_0 \in \partial\Omega$, we find another point $\omega_1 \in \Omega$ satisfying

$$e^{g_\Omega(\omega_1)} < q. \tag{25}$$

From Condition (3), (24), and (25) we get, for all sufficiently large n,

$$|Q_n(z_1)|^{1/n} > q > |Q_n(\omega_1)|^{1/n}.$$

It follows that Condition (5) cannot hold if F contains ω_1 and O contains z_1; note again that z_1 can be chosen arbitrarily close to $\partial\Omega$, and consequently an arbitrary open set O, $K \subset O$, contains a point with the property (24).

5.5. *Proof of Proposition* 1. Without loss of generality we assume that the point $z=0$ is an isolated point of K, and, as usual, we assume that $\operatorname{cap}(K)=1$. Let $\{Q_n(z)\}_1^\infty$ be a sequence of polynomials satisfying Condition (1) and, consequently, Conditions (2)–(4) as well, such that all the zeros of the polynomials $Q_n(z)$ are supported by K. Denote by $m=m(n)$ the number of zeros of $Q_n(z)$ at $z=0$, that is $Q_n(z)=\tilde{Q}_{n-m}(z)\cdot z^m$, where $\tilde{Q}_{n-m}(0)\neq 0$. Note that

$$m(n)/n \to 0 \qquad \text{as} \quad n \to \infty, \tag{26}$$

since otherwise the set $\{\mu_n\}$ of zero counting measures of $\{Q_n(z)\}$ would have a weak star limit μ_0 with a mass point $z=0$. But we know that if $\{Q_n(z)\}$ provides convergence, the sweeping out of μ_0 must be the equilibrium measure of K, and, consequently, μ_0 cannot have a mass point on the outer boundary of K.

Let us denote by $\tilde{\mu}_n$ the mass obtained from μ_n by omitting the mass at the point $z=0$. Because of (26) the set $\{\tilde{\mu}_n\}$ has the same weak star limits as the set $\{\mu_n\}$. From this it follows that we have

$$\lim_{n\to\infty} |\tilde{Q}_{n-m}(z)|^{1/n} = e^{g_\Omega(z)},$$

locally uniformly in the domain $\Omega \cup \{0\}$. In particular,

$$\lim_{n \to \infty} |\tilde{Q}_{n-m}(0)|^{1/n} = e^{g_\Omega(0)},$$

where $e^{g_\Omega(0)} = q > 1$ because of the irregularity of the point $z = 0$.

From the last limit relation we get

$$|\tilde{Q}_{n-m}(0)|^{1/n} > q - \varepsilon, \tag{27}$$

for an arbitrary $\varepsilon > 0$ and sufficiently large n.

We introduce $M = \sup_{n \in N} m(n)$ and consider two cases, $M < \infty$ and $M = \infty$. In the first case the polynomials $\{Q_n(z)\}$ can not provide convergence for the function $f(z) = 1/z^{M+1}$ because, due to the argument principle, the rational functions $P_n(z)/Q_n(z)$ must have at least $M+1$ poles close to the point $z = 0$, but they have at most M poles.

Consider now the case $M = \infty$. We choose a sequence of increasing indices $\{n_j\}_1^\infty$ so that for the corresponding indices $\{m_j\}_1^\infty$, $m_j = m(n_j)$, we also have $m_1 < m_2 < \cdots$. We introduce the function

$$f(z) = \sum_{i=1}^{\infty} \frac{a_{m_i}}{z^{m_i+1}},$$

where we shall specify the signs of a_{m_i} later and

$$|a_{m_i}| = |1 - \varepsilon|^{n_i}, \qquad 0 < \varepsilon < 1. \tag{28}$$

We have, due to (26),

$$|a_{m_i}|^{1/m_i} = |1 - \varepsilon|^{n_i/m_i} \to 0,$$

and, consequently, $f(z)$ is a holomorphic function everywhere in $\bar{\mathbb{C}} \backslash \{0\}$.

Consider the nonpolynomial part of $Q_{n_j}(z)\, f(z)$:

$$\begin{aligned}
\mathcal{O}(z^{-1}) &= Q_{n_j}(z)\, f(z) - P_{n_j}(z) \\
&= \tilde{Q}_{n_j - m_j}(z) \cdot z^{m_j} \left(\sum_{i=1}^{\infty} \frac{a_{m_i}}{z^{m_i+1}} \right) - P_{n_j}(z) \\
&= \tilde{Q}_{n_j - m_j}(z) \cdot z^{m_j} \left(\frac{a_{m_j}}{z^{m_j+1}} + \sum_{i \neq j} \frac{a_{m_i}}{z^{m_i+1}} \right) - P_{n_j}(z) \\
&= \frac{a_{m_j} \cdot \tilde{Q}_{n_j - m_j}(z)}{z} + \tilde{Q}_{n_j - m_j}(z) \cdot z^{m_j} \left(\sum_{i \neq j} \frac{a_{m_i}}{z^{m_i+1}} \right) - P_{n_j}(z).
\end{aligned}$$

Note that

$$\frac{\tilde{Q}_{n_j-m_j}(z)}{z}=T_j(z)+\frac{\tilde{Q}_{n_j-m_j}(0)}{z},$$

where $T_j(z)$ is some polynomial. This means that we can write

$$Q_{n_j}(z)\,f(z)-P_{n_j}(z)=\frac{a_{m_j}\cdot\tilde{Q}_{n_j-m_j}(0)}{z}+h_j(z),$$

where $h_j(z)$ is the singular part of the Laurent expansion of $Q_{n_j}(z)\,f(z)$ which does not depend on a_{m_j}. Now, fix a point $z_1\in\Omega$ and choose $a_{m_j}=|1-\varepsilon|^{n_j}$ or $a_{m_j}=-|1-\varepsilon|^{n_j}$ to guarantee that

$$\left|\frac{a_{m_j}\cdot\tilde{Q}_{n_j-m_j}(0)}{z_1}+h_j(z_1)\right|\geqslant\left|\frac{a_{m_j}\cdot\tilde{Q}_{n_j-m_j}(0)}{z_1}\right|.$$

By using (27) and (28) we get from the last inequality

$$|Q_{n_j}(z_1)\,f(z_1)-P_{n_j}(z_1)|\geqslant\frac{|1-\varepsilon|^{n_j}\,q-\varepsilon|^{n_j}}{|z_1|}\geqslant\frac{(q-(1+q)\varepsilon)^{n_j}}{|z_1|}. \tag{29}$$

Now assume that the point $z_1\in\Omega$ is sufficiently close to some regular point of K. Then, due to the continuity of the Green function at regular points we have $e^{g_\Omega(z_1)}<1+\varepsilon$. From the following property

$$\lim_{n\to\infty}|Q_n(z)|^{1/n}=e^{g_\Omega(z)}\qquad\text{in }\Omega$$

we obtain

$$|Q_n(z_1)|<|1+\varepsilon|^n \tag{30}$$

for all sufficiently large n. From (29) and (30) we get

$$\left|\frac{Q_{n_j}(z_1)\,f(z_1)-P_{n_j}(z_1)}{Q_{n_j}(z_1)}\right|\geqslant\frac{(q-(1+q)\varepsilon)^{n_j}}{|z_1|\cdot|1+\varepsilon|^{n_j}}.$$

If we choose ε small enough, the right-hand side of the last inequality tends to infinity with n_j.

Proposition 1 is proved.

5.6. *Proof of Corollary* 2. Let γ_n be positive numbers such that $\gamma_n Q_n$ has L^2_μ-norm 1. Since μ is a regular measure satisfying $\partial\Omega\subset S(\mu)\subset K$ and $\operatorname{cap}(K)=1$ we get ([10], Theorem 3.1.1, (i))

$$\lim_{n\to\infty}\gamma_n^{1/n}=1.$$

If we combine this with Corollary 1.1.5 in [10] and Remark 1 in this paper we obtain

$$\liminf_{n \to \infty} |\tilde{Q}_n(z)|^{1/m(n)} \geqslant e^{g_\Omega(z)}$$

uniformly on any given compact subset F of V. By using Theorem 3.2.1, (iii) in [10] and Remark 1 we get an inequality in the other direction:

$$\limsup_{n \to \infty} |\tilde{Q}_n(t)|^{1/m(n)} \leqslant e^{g_\Omega(t)}$$

uniformly on a contour $\gamma \in \Omega$ surrounding K. The desired result now follows from the error formula (1).

5.7. *Proof of Proposition* 3. If $K = \bar{\mathbb{C}} \backslash \Omega$ is convex, Theorem 1, (1) $\Rightarrow$ (3) shows that we cannot omit the condition that μ is regular.

Now we show that for irregular domains Ω we do not necessarily have the convergence result in Corollary 2 even if the measure μ is regular. We take $K = [-2, -1] \cup [1, 2] \cup \{0\}$, and the measure μ, $S(\mu) = K$, such that the restriction of μ to $K \backslash \{0\}$ is the linear Lebesgue measure; the regularity of μ easily follows from the Erdős–Turan Criterion (see [10], Corollary 4.1.2). In addition, μ is a symmetric measure and, consequently, all the zeros of the orthogonal polynomials Q_n lie on $K \backslash \{0\}$, except for one zero, in case of odd n, lying at $z = 0$. By taking $f(z) = 1/z^2$ and using arguments analogous to those used in the proof that Condition (2) does not imply Condition (1), we obtain that the corresponding PTAs do not converge to f in $\mathbb{C} \backslash K$. Proposition 3 is proved.

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